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Structure of gem-graphs

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Abstract The purpose of this paper is to introduce the gem-graphs, which are the natural extensions of trees and higher-dimensional trees (such as k-trees and k- trees) and determine their basic properties. We obtain a characterization of gem-graphs over paths of length at least 4 or cycles of length at least 5, and also determine their connectivity, centrality, planarity and Hamiltonian -property.

Index Terms— Keywords: Graph, Path, Cycle, Tree, Coloring

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1. Introduction

Higher-dimensional trees were first introduced by Harary and Palmer [6]. Later, the variation of these families of graphs were developed systematicallylly and studied in detail (see, Dewdney [5], Rose [15], Beineke et al. [2], Borowiecki et al. [3] and Patil et al. [10]. While trees are usually defined as those graphs which are connected and acyclic. This class of graphs can be equivalently defined by the following recursive construction rule:

Step 1. A single vertex (K_1) is a tree.

Step 2. Any tree T of order n ≥ 2, can be constructed from a tree Q of order (n − 1) by inserting an nth - vertex, and joining it to any vertex of Q.

Now, the aim of this paper is to extend the above tree- construction rule by allowing the base of the recursive growth (i.e., Step 1) to be any graph. With this view, we introduce and study the new family of graphs, whose recursive growth just starts from any given graph H. This kind of graphs, we simply call the gem-graphs over H.

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Definition1.1. The gem-graph G over the given graph H, we denote by G < H >, is the graph that can be obtained by the following recursive construction rule:

1. Given graph H of order k ≥ 1, is the smallest gem-graph.

2. To a gem - graph G < F > of order n ≥ k, insert a new $(n + 1)^{th}$ vertex, and join it to any set of k vertices : $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ of G < F >, so that the induced subgraph < $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ > is isomorphic to H.

It will be convenient to refer to the gem-graph G over H, by simply G<H>-graph. Sometimes, we shall abbreviate "gem-graph G over H" to simply, gem-graph G<H>. By the definition of the gem-graphs, the following facts are evident:

1. G < K_1>- graphs are th trees
2. G < K_2>- graphs are 2-trees, which are maximal outer planar graphs and studied in great detail (see,[11],[12]).

3. G < K_3>- graphs are 3-trees, which are the special family of maximal planar graphs, and studied in (see, [1], [7]).
4. G <K_k>- graphs are k-trees, which are extensively studied by many authors (see, [9], [13], [14]).
5. G <K_k>- graphs are k- trees, studied recently in (see,[10]).

However, we notice that the gem-graphs are the natural extensions of trees, k-trees and k-ctrees. In this paper, we study the elementary properties, and structural characterizations of the gem-graphs over paths P_k (for k ≥ 4) and cycles C_n (for n ≥ 5). In addition, we determine their connectivity, centrality, planarity, Hamiltonian property in great detail.

For a vertex v of a graph G, a neighbour of v is a vertex adjacent to v in G. The neighbourhood N (v) of v is the set of all neighbours of v. Thus, deg (v) = |N (v)|.

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1. Gem-graphs over a given graph

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The following theorem is simply a re-statement of the definition of gem-graphs.

Theorem 2.1. Let H be any graph of order k ≥ 1. Then a graph G of order p ≥ k + 1, is a gem-graph over H if and only if the following three conditions hold:

1. (G) can be labelled as $v_1, v_2, \dots, v_p$ so that for each positive integer i (k + 1 ≤ i ≤ p), there exist k distinct unordered labels: $i_1, i_2, \dots, i_k$ such that $< \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} > =$ H in G.
2. The induced subgraph $< \{v_{i_1}, v_{i_2}, \dots, v_{i_k}, v_i\} > =$ H + K_1 in G.
3. deg (v_i) = k in $< \{v_{i_1}, v_{i_2}, \dots, v_{i_k}, v_i\} >$.

Roughly speaking, G can be reduced to the base-graph H, by repeated removal of a vertex of degree k.

Definition 2.1. Let H be any graph of order k ≥ 1, and let G < H > be a gem-graph of order p ≥ k + 1. Then a vertex v of G < H > is called a H-vertex if all its neighbouring vertices in G induce the graph H.

Next, we present the structural characterization of a graph to be gem-graph over any given graph.

Theorem 2.2. Let H be any graph of order k ≥ 1. Then a graph G of order p ≥ k + 1, is a gem-graph over H if and only if G has a H-vertex v of degree k, and G − v is again a gem-graph over H.

By the repeated application of Theorem 2.2 to the gem-graphs, we have the following result.

Theorem 2.3. Let H be any graph of order k ≥ 1, and let G < H > be a gem-graph of order p ≥ k. Then

1. |E (G)| = |E (H)| + k (p − k).
2. If p ≥ k + 2, then G contains a subgraph isomorphic to H + $2K_1$.
3. If H contains m edges and t triangles, then the number of triangles in G < H > is t + m (p − k).

4. Characterization and properties of gem-graphs over paths or cycles

Notice that a gem-graph G < H >, where H is either a path P_3 or a cycle C_4, has highly irregular complex structure by nature, and its characterization appears to be hard in this case. The purpose of this section is to obtain a characterization and the properties of the gem-graphs G < H >, when H is either a path P_k (for k ≥ 4) or a cycle C_m (for m ≥ 5).

Theorem 3.1. Let G < H > be a gem-graph of order p ≥ k + 1, where H is either P_k for k ≥ 4 or C_k f or k ≥ 5. Then G< H > is isomorphic to H + (p − k) K_1.

Proof: Suppose H is a path P_k for k ≥ 4. We prove the result by induction on p.

If p = k + 1, then by definition, G <P_k> = P_k + K_1, which is obviously true.

Assume the result is true for any m < p. Next, we consider a gem-graph G over H, where H is P_k, having its order p. Let v be any H-vertex of G. By Theorem 2.2, G−v is a gem-graph of order p − 1. By induction hypothesis, we have G – v = H + (p−k−1) K_1. Consequently, G−v is the sum of two disjoint graphs: H = P_k, and I = (p−k−1) K_1.

Suppose v is adjacent to each vertex of H in G, then the result follows immediately. Otherwise, v is adjacent to at least one vertex of I in G. Moreover, deg (v) = k in G, we have two nonempty sets: A and B such that A ⊆ H; B ⊆ I, with A ∪ B = N (v), and |A| + |B| = k.

We discuss four cases depending on the cardinality of A and B:

Case1. |A| = k − 1, and |B| = 1. Since k ≥ 4,

Case 2. |A| = k − 2, and |B| = 2. Immediately, we have |A| ≥ 2 (because k ≥ 4).

We discuss two possibilities:

2.1. Suppose A is independent. Certainly, there are two non-adjacent vertices x and y in A. Let us consider B = {a, b}. Immediately, we see that < {x, y, a, b} > = C_4 appears in N (v), which is impossible.

2.2. Suppose A is non-independent. Then <A> contains at least one edge. In this situation, Case 1 repeats.

Case3. |A| = 1, and |B| = k − 1. It is easy to see that < N (v) > is a star-graph $K_{1,k−1}$, and is not possible.

Case4. |A| ≥ 2, and |B| ≥ 3.

There are two possibilities, depending on A:

4.1. Suppose A is non-independent. Then Case 1 repeats.

4.2. Suppose A is independent. Then Case 2 repeats.

In each of the above cases, we see that < N (v) > = H, and hence v is not a H-vertex of G. This is a contradiction. Therefore, v cannot be adjacent to any vertex of I.

Finally, assume H is a cycle C_k for $k \geq 5$. An analogous proof; that is, replacing P_k by C_k in the above arguments shows that G $<C_k>$ is isomorphic to $C_k + (p - k) K_1$.

A graph is self- centered if it is isomorphic to its center. It is well-known that for any hamiltonian graph G, and for every nonempty proper subset S of V (G), the number of components $\omega (G \setminus S) \leq |S|$.

Now, we study mainly the centrality, hamiltonian property, planarity and outerplanarity of gem-graphs over paths P_k (for $k \geq 4$) or a cycle C_m (for $m \geq 5$). We use the symbols, e(G) and t(G) denote the number of edges and triangles in a graph G, respectively and $\chi(G)$ denotes the chromatic number of G.

Proposition 3.2. Let G < H > be a gem-graph of order $p \geq k + 1$, where H is either a path P_k for $k \geq 4$ or a cycle C_k for $k \geq 5$.

1. (a). $e(G <P_k>) = k(p - k + 1) - 1$.
 (b). $e(G <C_k>) = k(p - k + 1)$.
2. (a). $t(G <P_k>) = (k - 1)(p - k)$.
(b). $t(G <C_k>) = k(p - k)$.

3. (a). $\chi(G <P_k>) = 3$.
(b). $\chi(G <C_k>) = \begin{cases} 3 & if \ \ k \ is \ even \\ 4 & if \ \ k \ is \ odd \end{cases}$
4. G < H > is a self- entered graph.
5. G < H > is hamiltonian if and only if $p \leq 2k$.
 (a). G < H > is planar if and only if $p \leq k + 2$.
 (b). G $<P_k$)> is outerplanar if and only if $p = k + 1$.
(c). G $<C_k$)> is non-outerplanar if and only if for all p : $(k + 1 \leq p \leq k + 2)$.

Proof: (1) and (2), directly follows from the definition of gem-graph. Next, we prove (3), G < H > contains at least one triangle by (2), and hence, it is not 2-colorable. This implies that χ (G<H>) ≥ 3. To achieve an upper bound, we shall produce a proper 3- coloring of G $<P_k>$, and also a proper (3 or 4)- coloring of G $<C_k>$. Since G<H> = $H + (p - k) K_1$.

Now, we discuss two cases depending on the nature of H.

3(a). When $H = P_k$ for $k \geq 4$. In this case, color the vertices of P_k by using colors 1 and 2. Further, assign the color 3 to all the vertices of $(p - k) K_1$. This completes a proper 3- coloring of G < P_k >.

3(b). When H = Ck for $k \geq 5$. Color the vertices of Ck by using two colors: 1 and 2, alternatively if k is even. Otherwise, use only three colors (1, 2, and 3). If k is even, then assign the color 3 to the vertices of $(p - k) K_1$. If k is odd, then assign the color 4 to the vertices of $(p - k) K_1$. Thus, a proper (3 or 4)- coloring of G $<C_k>$ is completed.

To prove (4), let us consider H is either P_k (for $k \geq 4$) or C_k (for $m \geq 5$). For every pair of vertices (u, v) in G < H >, where G<H> = $H + (p - k) K_1$, $u \in V (H)$ and $v \in V (H)$, the eccentricity e (u) = e (v) = 2. Consequently, G< H > is a self-centered graph.

For (5), assume that G< H > is hamiltonian and $p \geq 2k + 1$. Since $|V (H)| = k$, we have $|(p - k) K_1| = k + 1$. Consider S = V (H). Then $G \setminus S = (p - k) K_1$ and hence $\omega (G \setminus S) \geq k + 1$. This implies that G is not hamiltonian. So, $p \leq k + 1$.

To prove the converse, it is sufficient to obtain a Hamilton- cycle in G<H> = $H + tK_1$ for $1 \leq t \leq k$. Let V (H) = $\{u_1,u_2,\ldots ,u_k\}$, and V $(tK_1) = \{v_1,v_2,\ldots ,v_t\}$. Sin e $k \geq t$, we have $(k - t) = m \geq 0$. Immediately, the Hamilton- cycle: $u_1 u_2 \ldots u_{m+1} v_1 u_{m+2} v_2 u_{m+3} \ldots v_{t-1} u_k v_t u_1$ appears in G. Hence G<H> is Hamiltonian.

For 6(a), assume that G < H > is planar and $p \geq k + 3$. Immediately, we observe that $(H + 3K_1) \subseteq$ G< H >. Since $K_{3,3}$ appears as an induced subgraph in $(H + 3K_1)$, it follows that $K_{3,3}$ appears as a forbidden subgraph in G < H >, and hence by Kuratowski theorem G < H > is not planar. This is a contradiction to our assumption. Hence, $p \leq k + 2$. It is easy to prove the converse.

Finally, 6(b) and 6(c) are the immediate consequence of 6(a).

For any graph G, i (G) denotes the number of induced 4- cycles in G. Now, we determine this for gem-graphs.

Proposition 3.3.

Let G < H > be a gem-graph of order $p \geq k + 2$, where H is either P_k for $k \geq 4$ or C_k f or $k \geq 5$.

1. (a). $G\langle P_k \rangle$ is induced C_n-free for $n \geq 5$.
 (b).
$i(G < P_k) = \frac{1}{4}(p - k)(p - k - 1)(k^2 - 3k + 2)$

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$i(G < P_k) = \frac{1}{4}(p-k)(p-k-1)(k^2 - 3k + 2)$
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2. (a). $G\langle C_k\rangle$ is induced C_n-free for n ≥ 5and $n \neq k$.

(b). $i(G < C_k) = \frac{k}{4}(p-k)(p-k-1)(k-3)$.

Proof: In the proof of Theorem 3.1, we notice that the induced subgraphs resulted in a gem-graph G<H> = H + (p − k) K_1, are the only triangles, (4 or k)-cycles, stars and paths. With this, G< H > is certainly an induced C_n-free graph for n ≥ 5 and further, only in the case, when H = C_n (for n ≥ 5), then $n \neq k$.

To compute the number of 4-cycles in G<P_k>, notice that any 4-cycle in G<P_k> is resulted by choosing any two non-adjacent vertices from P_k and any two vertices from I, where I = (p − k) K_1. Moreover, we see that the number of pairs of non-adjacent vertices in P_k is $\left[\binom{k}{2} - (k-1)\right]$, and the number of 2-element subsets of I is $\binom{p-k}{2}$. Hence

$$i(G < P_k) = \left|\left(\binom{k}{2} - (k-1)\right).\binom{p-k}{2}\right| = \frac{1}{4}(p-k)(p-k-1)(k^2-3k+2).$$

Finally, to compute the number of 4-cycles in $G\langle C_k\rangle$, notice that a 4-cycle in $G\langle C_k\rangle$ is resulted by selecting any two non-adjacent vertices from C_k and any two vertices from I. Also, the number of pairs of non-adjacent vertices in C_k is $\left(\binom{k}{2} - k\right)$. Therefore,

$$i(G < C_k) = \left|\binom{k}{2} - k\right|.\binom{p-k}{2} = \frac{k}{4}(k-3)(p-k)(p-k+1)$$

Let $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and edge-connectivity of a graph G, respectively. It is well-known (due to Whitney) that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. Next, we study the connectivity of a gem- graph G < H >, when H is either P_k (for k ≥ 4) or C_k (for k ≥ 5).

Proposition 3.4.
Let G<P_k> for k ≥ 4
1. δ(G < H >) = k.
2. G<H > is k- connected (resp. k-edge-connected)

Proof: (1) follows directly by the recursive definition of gem-graphs. For (2), we notice that by removal of any (k −1) vertices from G < H > results in a connected graph, but removal of |V (H)| = k vertices from G < H > results in a disconnected graph. This shows that G< H > is k-connected. On the other hand, by Whitney theorem with (1) implies that κ′(G<H>) = k. Consequently, G< H > is k-edge- connected.

4. Open problems

.In Section 3, we have characterized only the gem-graphs G < H > for which, H is either a path P_k for k ≥ 4 or a cycle C_m for m ≥ 5. Now, the following problems remain open for further work: Characterize the class of gem-graphs G < P_3 > or G < C_4 >.

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